

partly answered by surveys of alternative theories of preference in risky situations (Chapter 3) and in uncertain situations (Chapter 8). These five chapters (1–3, 7–8) comprise a book within the book whose purpose is to provide an up-to-date view of expected utility and its alternatives that aims for organizational clarity and inclusiveness.

The other four chapters expand on question 3 by detailing new theories that I have had some role in developing. These chapters form a second book within the book that organizes material previously available only in a scattered set of journal articles.

The book is intended for graduate students and research workers in mathematics, economics, statistics, operations research, psychology, and related fields that are interested in the foundations and potential applications of decision making under risk and uncertainty.

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# Nonlinear Preference and Utility Theory

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## Preface

The theories of John von Neumann and Oskar Morgenstern for preference comparisons under risk and of Frank P. Ramsey and Leonard J. Savage for preference comparisons under uncertainty have been widely adopted as the quintessential paradigms for rational decision making in the face of uncertainty. Their expected (linear) utility models have profoundly affected economic analysis, risk assessment, and statistical decision theory over the past 35 years.

During this period there has been a growing awareness—stimulated in large part by Maurice Allais's work in the early 1950s—that people's reasoned judgments often violate the basic assumptions of expected utility. Numerous studies have demonstrated that such violations tend to follow systematic and predictable patterns. Investigators have therefore proposed alternative theories of rational preference that accommodate systematic departures from expected utility while retaining much of its mathematical elegance and computational convenience. With the notable exception of Allais's own theory from the early 1950s, most of the alternatives have emerged during the past decade.

This book attempts to make sense of recent and not so recent developments in preference theory for risky and uncertain decisions. It is arranged in two main parts: Chapters 1 through 6 focus on decision under risk, Chapters 7 through 9 on decision under uncertainty. Each main part answers three questions:

1. What is expected utility theory?
2. What is wrong with it from a rational preference perspective?
3. What has been proposed to correct the problems?

Question 1 is dealt with in Chapter 1 for decision under risk (von Neumann and Morgenstern) and in Chapter 7 for decision under uncertainty (Savage). Chapter 2 and the first part of Chapter 8 address question 2. Question 3 is